On the Failure of the Linkage Principle with Colluding Bidders - Webappendix*

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1 Guide to the webappendix

This note contains additional material for Seres (2016). Section A motivates the argument that lower variance of demand decreases equilibrium price in Bertrand competition. The following Section B provides the example for the unique symmetric BNE of the auction when bidders act in a non-cooperative way. Section C provides the reduced-form of the incentive compatibility constraints and the respective Mathematica code. Section D contains derivations for the expected revenue result.

A Lower variance of demand reduces market price

Let us consider a discrete model following Tirole (1989). There are two firms engaging in price competition. Their profit is

$$\Pi_i\left(p_i, p_j\right) = \left(p_i - c_i\right) D\left(p_i, p_j\right)$$

where p_i and p_j are prices and c_i is production cost. Producers have $c_i \in \{\underline{c}, \bar{c}\}$ independently distributed, with equal probabilities, and this is private

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information. Demand can have a low or high slope with equal probabilities. Formally,

$$D\left(p_i, p_j\right) = \begin{cases} a - bp_i, & \text{if } p_i \leq p_j; \\ a - bp_j, & \text{if } p_j \leq p_i; \end{cases}$$

where $b \in \left\{\underline{b}, \bar{b}\right\}$, with equal probabilities, $\underline{b} = b^* - \epsilon$, $\bar{b} = b^* + \epsilon$, $\epsilon \geq 0$ and $b \geq \frac{1}{2}a$. That is, higher ϵ denotes more uncertainty about demand. Producers receive a noisy signal b_i about the realization of b. Formally, $Prob\left(b_i = b\right) = \delta \in \left(\frac{1}{2}, 1\right)$.

Let us show that equilibrium bids are increasing in δ. We demonstrate this for the low value signals $(\underline{c}, \underline{b})$, the other cases are analogous. The RHS of equation (1) is the expected pay-off of a player, conditionally on the opponent having the same type. We point out that this value is zero in BNE. Suppose it is positive. In that case, the competitor find it profitable to deviate if her values are also $(\bar{c}, b^* + \epsilon)$ by marginally lower her price. [1]

$$0 = \frac{1}{2}\left(Pr\left(b = \bar{b}|b_i\right)\left[\left(a + \bar{c}\bar{b}\right)p_i - a\bar{c} - \bar{b}p_i^2\right] + Pr\left(b = \underline{b}|b_i\right)\left[\left(a + \bar{c}\underline{b}\right)p_i - a\bar{c} - \underline{b}p_i^2\right]\right) \tag{1}$$

We can solve for p_i. There is a single positive root. If we take the first-order derivative of the solution with respect to ϵ, we get

$$2\left(a + \bar{c}b + (2Pr\left(b = \bar{b}|b_i\right) - 1)\epsilon\right) - 4a\bar{c} > 0 \tag{2}$$

since $Pr\left(b = \bar{b}|b_i\right) \geq \frac{1}{2}$. We can obtain the same result for all types. Thus, lower uncertainty (lower ϵ) decreases market price.

[1] We can derive the same result analogously for information sets $\left(\underline{c}, \bar{b}\right)$, $(\bar{c}, \underline{b})$ and $\left(\bar{c}, \bar{b}\right)$. If a type can achieve a positive pay-off, the competitor with identical type can marginally reduce price and obtain higher expected pay-off, since her pay-off makes a discrete upward jump with identical values and only changes marginally or does not change for other types. That is, all types have zero expected profit conditional on identical opposing types in equilibrium.

B Non-cooperative equilibrium with two ring members and one outsider

We can point out, that there exists a symmetric equilibrium in the game defined in the article in which players submit the following.[2] There are two cases, depending on the ranking of equilibrium strategies $b^*(1,z)$ and $b^*(0,1-z)$. In case the first one is higher:

$$b^*(x_i,y_i)=\begin{cases}\frac{3-5\delta+9\delta^2-\delta^3}{3-5\delta+5\delta^2}+\frac{3-5\delta-3\delta^2+2\delta^3}{3-5\delta+5\delta^2}z, & \text{if } x_i=1,\ y_i=1-z;\\ \frac{2-6\delta+6\delta^2-\delta^3}{1-3\delta+3\delta^2}+\frac{-1+3\delta-3\delta^2+2\delta^3}{1-3\delta+3\delta^2}z, & \text{if } x_i=1,\ y_i=z;\\ \frac{2\delta^2-\delta^3}{1+\delta^2-\delta^3}+\frac{1-\delta^2}{1+\delta^2-\delta^3}z, & \text{if } x_i=0,\ y_i=1-z;\\ \frac{1-3\delta+3\delta^2-\delta^3}{1-3\delta+3\delta^2}+\frac{\delta^3}{1-3\delta+3\delta^2}z, & \text{if } x_i=0,\ y_i=z;\end{cases}$$

On the other hand, if $b^*(1,z)<b^*(0,1-z)$, then:

$$b^*(x_i,y_i)=\begin{cases}\frac{3-5\delta+9\delta^2-\delta^3}{3-5\delta+5\delta^2}+\frac{3-5\delta-3\delta^2+2\delta^3}{3-5\delta+5\delta^2}z, & \text{if } x_i=1,\ y_i=1-z;\\ \frac{6-14\delta+12\delta^2-\delta^3}{3-7\delta+7\delta^2}+\frac{-3+7\delta-3\delta^2+2\delta^3}{3-7\delta+7\delta^2}z, & \text{if } x_i=1,\ y_i=z;\\ \frac{2\delta^2-\delta^3}{1-\delta+\delta^2}+\frac{1-\delta-3\delta^2+2\delta^3}{1-\delta+\delta^2}z, & \text{if } x_i=0,\ y_i=1-z;\\ \frac{1-3\delta+3\delta^2-\delta^3}{1-3\delta+3\delta^2}+\frac{\delta^3}{1-3\delta+3\delta^2}z, & \text{if } x_i=0,\ y_i=z;\end{cases}$$

We can note that there is no profitable deviation from this equilibrium. Every bid is determined in such a way, that the bidder is indifferent between obtaining the commodity or not, if the highest opposing bid is identical.

As an example, consider the highest value, which is characterized by $x_i=1$ and $y_i=1-z$. The result is the same for both cases above. Since the highest opposing bid is equal to $b^*(1,1-z)$, this is the amount the bidder has to pay. Indifference implies that this value equals the expected valuation of this bidder. The private value is determined by the signal, $x_i=1$. The common value can take two values *ex ante* with equal probabilities, $Pr(y=z)=Pr(y=1-z)=\frac{1}{2}$.

Players form Bayesian beliefs according to the signals. In the following we derive the conditional probability of $y=1-z$ if the highest opposing bid equals a player‘s bid. If the common value is high ($y=1-z$), the probability

[2] Derivations here refer to a version of the model with a normalization in which common value signals take values z and $1-z$ where $z\geq\frac{1}{2}$. The results are equivalent.

of a high signal is δ. Conditionally on that, the probability that the highest opposing bid is identical is derived as follows. The probability of the third highest bid being identical equals $\frac{1}{4}\delta^2$. If it is smaller, it can occur for both other bids. Having signals $(1, 1-z)$ gives $\frac{1}{2}\delta$. Having any other signal occurs with probability $1 - \frac{1}{2}\delta$. The sum is equal to $A \equiv \frac{1}{4}\delta^2 + 2\frac{1}{2}\delta\left(1 - \frac{1}{2}\delta\right)$.

Similarly, in case of a low common value $(y = z)$, the probability of a high signal equals $1 - \delta$. Conditionally on that, the highest opponent bid is identical with probability $B \equiv \frac{1}{4}(1-\delta)^2 + 2\frac{1}{2}(1-\delta)\left(1 - \frac{1}{2}(1-\delta)\right)$. Thus, we can get the expected revenue, which is equal to the expected payment, conditionally on an identical highest opening bid as follows. The probability that a player with signal $(1, 1-z)$ and observing an identical highest opposing value have a high common value $1-z$ can be derived from the conditional probabilities above as $\frac{\delta A}{\delta A + (1-\delta)B}$. The conditional probability of the opposite follows from that as $\frac{(1-\delta)B}{\delta A + (1-\delta)B}$. Thus, the equilibrium bid is the sum of the private value and the conditional expected common value in equation (3).

$$\begin{aligned} b^*(1, 1-z) &= 1 + \frac{\delta A}{\delta A + (1-\delta)B}(1-z) + \frac{(1-\delta)B}{\delta A + (1-\delta)B}z \\ &= \frac{3 - 5\delta + 9\delta^2 - \delta^3}{3 - 5\delta + 5\delta^2} + \frac{3 - 5\delta - 3\delta^2 + 2\delta^3}{3 - 5\delta + 5\delta^2}z \end{aligned} \tag{3}$$

If a higher bid is submitted, the expected pay-off remains zero. If the bid is lower, probability of winning yields zero, again resulting in zero pay-off.

C Two forms of the incentive compatibility constraint in second-price auctions

The incentive compatibility constraint (5) can take two forms, depending on the rank of non-cooperative equilibrium bids. Here we keep the number of ring members 2, and the number of outsiders 1. From the unique symmetric Bayesian equilibrium, we can formulate this according to the sign of q defined in equation (4), here repeated from the article.

$$q \equiv \frac{6 - 14\delta + 12\delta^2 - \delta^3}{3 - 7\delta + 7\delta^2} + \frac{-3 + 7\delta - 3\delta^2 + 2\delta^3}{3 - 7\delta + 7\delta^2}z - \left(\frac{2\delta^2 - \delta^3}{1 - \delta + \delta^2} + \frac{1 - \delta - 3\delta^2 + 2\delta^3}{1 - \delta + \delta^2}z\right) \tag{4}$$

Given parameters δ and z, there is an incentive compatible BCM if $c(\delta, z) \geq 1$, where

$$c(\delta, z) = \begin{cases} d(\delta, z), & \text{if } q \geq 0; \\ e(\delta, z), & \text{if } q < 0; \end{cases}$$

and $d(\delta, z)$, $e(\delta, z)$ are derived according to equations (6) and (7) which come from a substitution to the left-hand side of equation (5).

$$\theta \Pi(1, z, z) + (1-\theta) \Pi(1, z, 1-z) \geq \theta \Pi(0, 1-z, 1-z) + (1-\theta) \Pi(0, z, 1-z) \tag{5}$$

$$\begin{aligned} d(\cdot) = \Bigg[& \theta \left(\frac{1}{2} \left(a + 1 + \frac{\delta^2}{\delta^2 + (1-\delta)^2} z + \frac{(1-\delta)^2}{\delta^2 + (1-\delta)^2} (1-z) \right.\right. \\ & - \left.\left. \frac{\delta(1-\delta)}{\delta^2 + (1-\delta)^2} b^*(0, 1-z) - \frac{\delta^3 + (1-\delta)^3}{\delta^2 + (1-\delta)^2} b^*(0, z) \right) \right) \\ & + (1-\theta) \left(\frac{3}{4} \left(a + 1 + \frac{1+\delta}{2+\delta} z + \frac{1}{2+\delta} (1-z) - \frac{1}{3} \left(b^*(1, z) + b^*(0, 1-z) + b^*(0, z) \right) \right) \right) \Bigg] / \\ & \Bigg[\theta \left(\frac{1}{2} \left(a + \frac{\delta^2}{\delta^2 + (1-\delta)^2} (1-z) + \frac{(1-\delta)^2}{\delta^2 + (1-\delta)^2} z \right.\right. \\ & - \left.\left. \frac{\delta^3 + (1-\delta)^3}{\delta^2 + (1-\delta)^2} b^*(0, 1-z) - \frac{\delta(1-\delta)}{\delta^2 + (1-\delta)^2} b^*(0, z) \right) \right) \\ & + (1-\theta) \left(\frac{1}{4} \left(a + \delta z + (1-\delta)(1-z) - b^*(0, z) \right) \right) \Bigg] \end{aligned} \tag{6}$$

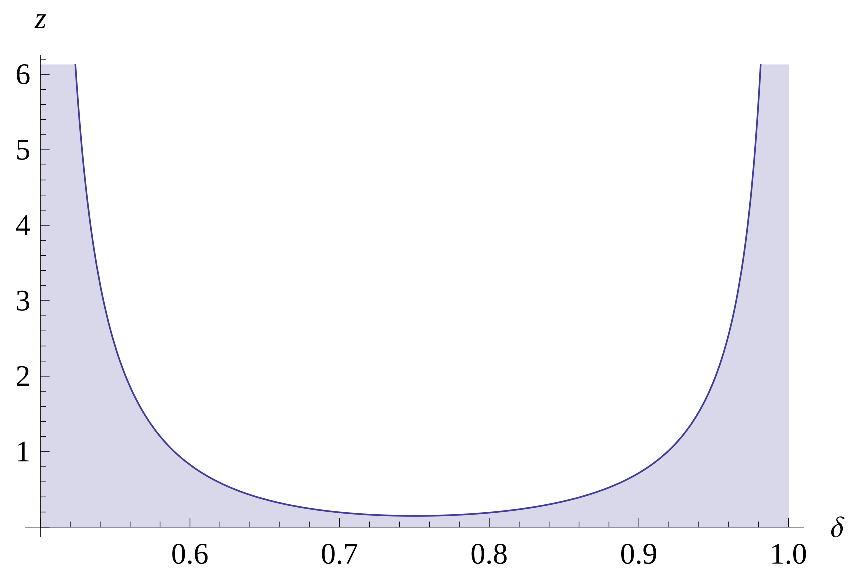

Figure 1: Existence of incentive compatible BCM, two ring members and one outsider

$$
\begin{aligned}
e(\cdot) = \Bigg[& \theta \left(\frac{\delta^3 + (1-\delta)^3}{\delta^2 + (1-\delta)^2} \left(a + 1 + \frac{\delta^3}{\delta^3 + (1-\delta)^3} z + \frac{(1-\delta)^3}{\delta^3 + (1-\delta)^3} (1-z) \right.\right. \\
& - \left.\left. \frac{1}{2} b^* (0, 1-z) - \frac{1}{2} b^* (0, 1-z) \right) \right) \\
& + (1-\theta) \left(\frac{3}{4} \left(a + 1 + \frac{1+\delta}{2+\delta} z + \frac{1}{2+\delta} (1-z) - \frac{1}{3} \left(b^* (1, z) + b^* (0, 1-z) + b^* (0, z) \right) \right) \right) \Bigg] / \\
& \Bigg[\theta \left(\frac{\delta (1-\delta)}{\delta^2 + (1-\delta)^2} \left(a + \frac{\delta^3}{\delta^3 + (1-\delta)^3} (1-z) + \frac{(1-\delta)^3}{\delta^3 + (1-\delta)^3} z \right.\right. \\
& - \left.\left. \frac{1}{2} b^* (0, 1-z) - \frac{1}{2} b^* (0, z) \right) \right) \\
& + (1-\theta) \left(\frac{1}{4} \left(a + \delta z + (1-\delta)(1-z) - b^* (0, z) \right) \right) \Bigg] \qquad (7)
\end{aligned}
$$

Accordingly, the set of incentive compatible BCM's resulting from the equations above is illustrated as the shaded area in Figure 1.

Mathematica codes are given below. δ and z are replaced by x and y, respectively. Code of solution for q:

```
Solve[(6-14*x+12*x^2-x^3)/(3-7*x+7*x^2)+(-3+7*x-3*x^2+2*x^3)/(3-7*x+7*x^2)

*y-((2*x^2-x^3)/(1-x+x^2)+(1-x-3*x^2+2*x^3)/(1-x+x^2)*y)==0,y]
```

Solution for $q(\cdot)$ with respect to z is

$$f_0(\delta) = \frac{(1-x)^2(3-4x+2x^2+3x^3)}{3-10x+9x^2+4x^3-15x^4+6x^5}$$

Code for $e(\cdot)$ is:

```
Solve[
((x^2+(1-x)^2)*((x^3+(1-x)^3)/(x^2+(1-x)^2)*(1+(x^3)/(x^3+(1-x)^3)*y+

(((1-x)^3)/(x^3+(1-x)^3))*(1-y)-1/2*(1+((1-x)^3)/((1-x)^3+x^3)*(1-y)+

(x^3)/((1-x)^3+x^3)*y)-1/2*((1-3*x+3*x^2-x^3)/(1-3*x+3*x^2)+

(3*x-1-3*x^2+2*x^3)/(1-3*x+3*x^2)*y)))+(1-x^2-(1-x)^2)*(3/4*(1+(2*x-1)

/3*y+(2-x)/3-1/3*(1+((1-x)^3)/((1-x)^3+x^3)*(1-y)+(x^3)/((1-x)^3+x^3)*y)

-1/3*((x^2*(1-1/2*x))/(x^2*(1-1/2*x)+1/2*(1-x^2))*(1-y)+(1/2*(1-x^2))

/(x^2*(1-1/2*x)+1/2*(1-x^2))*y)-1/3*((1-3*x+3*x^2-x^3)/(1-3*x+3*x^2)

+(3*x-1-3*x^2+2*x^3)/(1-3*x+3*x^2)*y))))/((x^2+(1-x)^2)*((x*(1-x))/

(x^2+(1-x)^2)*((x^3)/(x^3+(1-x)^3)*(1-y)+((1-x)^3)/(x^3+(1-x)^3)*y

-1/2*(1+((1-x)^3)/((1-x)^3+x^3)*(1-y)+(x^3)/((1-x)^3+x^3)*y)-1/2*

((1-3*x+3*x^2-x^3)/(1-3*x+3*x^2)+(3*x-1-3*x^2+2*x^3)/(1-3*x+3*x^2)*y))

-(x^2)(2-8*x+9*x^2-4*x^3+x^4)*(-1+2*y))/(2*(1-2*x+2*x^2)*(-1-x^2+x^3))

+(1-x^2-(1-x)^2)*(1/4*(x*y+(1-x)*(1-y)-((1-3*x+3*x^2-x^3)/
```

```
(1-3*x+3*x^2)+(3*x-1-3*x^2+2*x^3)/(1-3*x+3*x^2)*y))))==1,y]
```

Solution for $e(\cdot)$ is

$$f_1(\delta) = \frac{-1+\delta+2\delta^2+6\delta^3-35\delta^4+51\delta^5-31\delta^6+6\delta^7}{2\delta(-2+6\delta+2\delta^2-29\delta^3+48\delta^4-31\delta^5+6\delta^6)}$$

Code for $d(\cdot)$ is:

```
Solve[(((x^2+(1-x)^2))*(1/2*(1+(x^2)/(x^2+(1-x)^2)*y+((1-x)^2)/
(x^2+(1-x)^2)*(1-y)-(x*(1-x))/(x^2+(1-x)^2)*((2*x^2-x^3)/(1-x+x^2)
*(1-y)+(1-x-x^2+x^3)/(1-x+x^2)*y)-(x^3+(1-x)^3)/(x^2+(1-x)^2)*
((1-3*x+3*x^2-x^3)/(1-3*x+3*x^2)+(x^3)/(1-3*x+3*x^2)*y)))+
(1-(x^2+(1-x)^2))*(3/4*(1+(2*x-1)/3*y+(2-x)/3-1/3*(1+(3-7*x+5*x^2-x^3)
/(3-7*x+7*x^2)*(1-y)+(2*x^2+x^3)/(3-7*x+7*x^2)*y)-1/3*((2*x^2-x^3)
/(1-x+x^2)*(1-y)+(1-x-x^2+x^3)/(1-x+x^2)*y)-1/3*((1-3*x+3*x^2-x^3)
/(1-3*x+3*x^2)+(x^3)/(1-3*x+3*x^2)*y))))/((x^2+(1-x)^2)*(1/2*((x^2)
/(x^2+(1-x)^2)*(1-y)+((1-x)^2)/(x^2+(1-x)^2)*y-(x^3+(1-x)^3)/
(x^2+(1-x)^2)*((2*x^2-x^3)/(1-x+x^2)*(1-y)+(1-x-x^2+x^3)/(1-x+x^2)*y)-
(x*(1-x))/(x^2+(1-x)^2)*((1-3*x+3*x^2-x^3)/(1-3*x+3*x^2)+(x^3)/
(1-3*x+3*x^2)*y)))+(1-(x^2+(1-x)^2))*(-1-6*x^2*y+2*x*(1+y)+x^3*(-2+4*y)
/(4 (1-3*x+3*x^2))+1/4*(x*y+(1-x)*(1-y)-((1-3*x+3*x^2-x^3)/(1-3*x+3*x^2)
+(x^3)/(1-3*x+3*x^2)*y))))==1,y]
```

Solution for $d(\cdot)$ is

$$f_2(\delta) = \frac{-3 + 10\delta + 7\delta^2 - 86\delta^3 + 129\delta^4 + 86\delta^5 - 502\delta^6 + 664\delta^7 - 402\delta^8 + 94\delta^9}{-3 + 19\delta - 62\delta^2 + 186\delta^3 - 570\delta^4 + 1348\delta^5 - 2083\delta^6 + 1963\delta^7 - 1014\delta^8 + 216\delta^9}$$

Since $f_0(\delta) \geq f_1(\delta) \geq f_2(\delta)$ for any $\delta \in \left(\frac{1}{2}\right), 1$, we have that $f_1(\delta)$ determines the set of cut-off points as depicted on Figure 1.

D Expected revenue function

The collusive expected revenue depends on the sign of q. If it is positive, expected revenue equals to $E_1[R]$, if it is negative, it equals to $E_2[R]$, according to equations (8) and (9). We can note that both functions have a positive first-order derivative with respect to z, thus $0 < \frac{\partial E_1[R]}{\partial z}$ and $0 < \frac{\partial E_2[R]}{\partial z}$ if $\frac{1}{2} < x < 1$. In other words, the expected revenue is an increasing function of the publicly revealed information z on the continuous intervals of the domain.

$$\begin{aligned} E_1[R] = a + \frac{1}{8} &\left[\left(\delta^2 + (1-\delta)^2\right) b^*(1, 1-z)\right. \\ &+ \left(2\delta^2 + 2(1-\delta)^2 + 2\delta(1-\delta)\right) b^*(1, z) \\ &+ \left(2\delta^3 + 3\delta(1-\delta) + 2(1-\delta)^3 + 2\delta(1-\delta)\right) b^*(0, 1-z) \\ &+ \left.\left(\delta^3 + 4\delta(1-\delta) + 2\delta^2 + (1-\delta)^3 + 2(1-\delta)^2\right) b^*(0, z)\right] \end{aligned} \tag{8}$$

$$\begin{aligned} E_2[R] = a + \frac{1}{8} &\left[\left(\delta^2 + (1-\delta)^2\right) b^*(1, 1-z)\right. \\ &+ \left(3\delta^3 + 3\delta(1-\delta) + 3(1-\delta)^3 + 2\delta(1-\delta)\right) b^*(1, z) \\ &+ \left(\delta^3 + 4\delta(1-\delta) + 2\delta^2 + (1-\delta)^3 + 2(1-\delta)^2\right) b^*(0, 1-z) \\ &+ \left.\left(\delta^3 + 4\delta(1-\delta) + 2\delta^2 + (1-\delta)^3 + 2(1-\delta)^2\right) b^*(0, z)\right] \end{aligned} \tag{9}$$

The expected revenue function can be depicted with following code.

```
e[x_,y_]:=1/8*((x^2+(1-x)^2)*(1+(4*x^2-x^3)/(3-5*x+5*x^2)*(1-y)+
(3-5*x+x^2+x^3)/(3-5*x+5*x^2)*y)+(2*x^2+2*(1-x)^2+2*x*(1-x))*
(1+(3-7*x+5*x^2-x^3)/(3-7*x+7*x^2)*(1-y)+(2*x^2+x^3)/(3-7*x+7*x^2)*y)
+(2*x^3+3*(1-x)*x+2*(1-x)^3)*((2*x^2-x^3)/(1-x+x^2)*(1-y)+
(1-x-x^2+x^3)/(1-x+x^2)*y)+(x^3+4*x*(1-x)+2*x^2+(1-x)^3+2*(1-x)^2)*
((1-3*x+3*x^2-x^3)/(1-3*x+3*x^2)*(1-y)+(x^3)/(1-3*x+3*x^2)*y)))
```

```
e[x,y]/.x->0.6

f[x_,y_]:=
1/8*((x^2+(1-x)^2)*(1+(4*x^2-x^3)/(3-5*x+5*x^2)*(1-y)+
(3-5*x+x^2+x^3)/(3-5*x+5*x^2)*y)+(3*x^3+3*x*(1-x)+3*(1-x)^3+2*x*(1-x))
*(1+(3-7*x+5*x^2-x^3)/(3-7*x+7*x^2)*(1-y)+(2*x^2+x^3)/(3-7*x+7*x^2)*y)+
(x^3+4*x*(1-x)+2*x^2+(1-x)^3+2*(1-x)^2)*((2*x^2-x^3)/(1-x+x^2)*(1-y)+
(1-x-x^2+x^3)/(1-x+x^2)*y)+(x^3+4*x*(1-x)+2*x^2+(1-x)^3+2*(1-x)^2)
*((1-3*x+3*x^2-x^3)/(1-3*x+3*x^2)*(1-y)+(x^3)/(1-3*x+3*x^2)*y)))

f[x,y]/.x->0.6
```

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